(CLD, 22/09/05)

E173–Homework No. 3

1. What equation must the function $f(r)$ satisfy in order that the Airy stress function

$$\varphi(x,z) = f(r)\cos n\theta$$

satisfies the appropriate biharmonic compatibility equation?

2. Determine the (final) form of the differential equation for $f(r)$ that results from first writing that equation as the operator sequence

$$r\frac{d}{dr}\left\{\frac{1}{r^3}\frac{d}{dr}\left[r^3\frac{d}{dr}\left(\frac{1}{r^3}\frac{d\left(r^2 f(r)\right)}{dr}\right)\right]\right\} = 0$$

3. Determine the (final) expanded form of the differential equation for $f(r)$ that corresponds to

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{4}{r^2}\right)\left(\frac{d^2 f(r)}{dr^2}+\frac{1}{r}\frac{df(r)}{dr}-\frac{4f(r)}{r^2}\right) = 0$$

How does this result compare with those obtained in Problems 1 and 2?

4. For an infinitely long cylinder loaded only with an internal pressure p_i, is it possible for there to be a ring or circumference of radius r that remains unstrained (i.e., for which $\varepsilon_{\theta\theta} = 0$)? If the cylinder is loaded only with an external pressure p_o? If there is such a line in either case, determine the value of the radius at which it occurs. If not, explain your answer(s).

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1. $\nabla^2\left(\nabla^2(f(r)\cos n\theta)\right) =$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}\right)\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}\right)f(r)\cos n\theta$$

$$\Rightarrow \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}\right)\left(\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{n^2}{r^2}f\right)=0$$

After a bit of algebra:

$$\boxed{\frac{d^4f}{dr^4}+\frac{2}{r}\frac{d^3f}{dr^3}-\frac{2n^2+1}{r^2}\frac{d^2f}{dr^2}+\frac{2n^2+1}{r^3}\frac{df}{dr}+\frac{n^2(n^2-4)}{r^4}f=0}$$

2. $$r\frac{d}{dr}\left(\frac{1}{r^3}\frac{d}{dr}\left\{r^3\frac{d}{dr}\left[\frac{1}{r^3}\frac{d(r^2f)}{dr}\right]\right\}\right)=0$$

$$\Rightarrow \boxed{\frac{d^4f}{dr^4}+\frac{2}{r}\frac{d^3f}{dr^3}-\frac{9}{r^2}\frac{d^2f}{dr^2}+\frac{9}{r^3}\frac{df}{dr}=0}$$

3. $$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{4}{r^2}\right)\left(\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{4}{r^2}f\right)=0$$

$\Rightarrow$

3a. Exactly same result as Prob. 2.

3b. Same result as Prob. 1 w/ $n=2$.

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4. $\varepsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu\sigma_{rr}\right)$

$$= \frac{1-\nu}{E}\left(\frac{a^2p_i - b^2p_o}{b^2-a^2}\right)\left[1 - \left(\frac{1+\nu}{1-\nu}\right)\frac{b^2(p_o-p_i)}{(a^2p_i-b^2p_o)}\left(\frac{a}{r}\right)^2\right]$$

For $\varepsilon_{\theta\theta} = 0$, $\left(\frac{r}{a}\right)^2 = \left(\frac{1+\nu}{1-\nu}\right)\frac{b^2(p_o-p_i)}{(a^2p_i-b^2p_o)}$

① For $p_i = p$, $p_o = 0$: $\left(\frac{r}{a}\right)^2 = -\left(\frac{1+\nu}{1-\nu}\right)\left(\frac{b}{a}\right)^2 < 0$

$\therefore$ $\boxed{\varepsilon_{\theta\theta} \text{ can NOT vanish for } p = p_i}$

② For $p_i = 0$, $p_o = p$: $\left(\frac{r}{a}\right)^2 = -\left(\frac{1+\nu}{1-\nu}\right) < 0$

$\therefore$ $\boxed{\varepsilon_{\theta\theta} \text{ can NOT vanish for } p = p_o.}$

(Remember, $-1 \le \nu \le \frac{1}{2}$.)